Last time: Line integrals

Fundamental theorem of Line integrals

Give curve C parameterized by $\vec{r}(t)$ on $[a,b]$ and f a function w/ continuous partial derivatives. Then

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

Where C is oriented from $\vec{r}(a)$ to $\vec{r}(b)$

Recall switching the orientation of curve C negates the corresponding line integral. i.e $\int_{-C} \vec{v} \cdot d\vec{r} = -\int_C \vec{v} d\vec{r}$

Ex: compute $\int_C \vec{v} \cdot d\vec{r}$ for $\vec{v} = \langle \sin y,\ x\cos(y) + \cos z,\ -y\sin(z) \rangle$ and curve C parameterized by $\vec{r}(t) = \langle \sin(t), t, 2t \rangle$ on $\langle 0, \frac{\pi}{2} \rangle$

Sol: First we check $\vec{v}$ is conservative

$$\frac{\partial}{\partial y}[V_x] = \frac{\partial}{\partial y}[\sin(y)] = \cos(y)$$

$$\frac{\partial}{\partial z}[V_x] = \frac{\partial}{\partial z}[\sin(y)] = 0$$

$$\frac{\partial}{\partial x}[V_y] = \frac{\partial}{\partial x}[x\cos(y) + \cos z] = \cos(y)$$

$$\frac{\partial}{\partial z}[V_y] = \frac{\partial}{\partial z}[x\cos(y) + \cos z] = -\sin(z)$$

$$\frac{\partial}{\partial x}[V_z] = \frac{\partial}{\partial x}[-y\sin(z)] = 0$$

$$\frac{\partial}{\partial y}[V_z] = \frac{\partial}{\partial y}[-y\sin(z)] = -\sin(z)$$

$\therefore$ by a previous result $\vec{v}$ is conservative

i.e. $\vec{v} = \nabla f$ for some function f.

Next, we compute such a potential function.

$$\frac{\partial f}{\partial x} = \sin y \qquad \frac{\partial f}{\partial y} = x\cos(y) + \cos(z), \qquad \frac{\partial f}{\partial z} = -y\sin z$$

$$f(x,y,z) = \int \frac{\partial f}{\partial z}\, dz = \int -y\sin z\, dz = y\cos z + C(x,y)$$

$$\sin(y) = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[y\cos(z) + C(x,y)\right] = \frac{\partial C}{\partial x}$$

$$\therefore\ C(x,y) = \int \sin(y)\, dx = \int \frac{\partial C}{\partial x}\, dx = x\sin(y) + D(y)$$

hence $f(x,y,z) = y\cos(z) + C(x,y) = y\cos(z) + x\sin(y) + D(y)$

$$\therefore\ x\cos(y) + \cos(z) = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[y\cos(z) + x\sin(y) + D(y)\right]$$

$$= \cos z + x\cos(y) + D'(y)$$

$\therefore D'(y) = 0$ So $D(y) = E$ is constant.

$\therefore f(x,y,z) = y\cos(z) + x\sin(y)$ is a potential for $\vec{v}$ (setting $E=0$)

$\therefore$ we may express $\int_C \vec{v}\cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} \overset{FTLI}{=} f(\vec{r}(b)) - f(\vec{r}(a))$

Now $\vec{r}(b) = \vec{r}(\frac{\pi}{2}) = \langle \sin\frac{\pi}{2}, \frac{\pi}{2}, 2\cdot\frac{\pi}{2}\rangle = \langle 1, \frac{\pi}{2}, \pi\rangle$

and $\vec{r}(a) = \vec{r}(0) = \langle \sin 0, 0, 2\cdot 0\rangle = \langle 0, 0, 0\rangle$

Hence $\int_C \vec{v}\cdot d\vec{r} = f(1, \frac{\pi}{2}, \pi) - f(0,0,0)$

$$= \left(\frac{\pi}{2}\cos\pi + 1\cdot\sin\frac{\pi}{2}\right) - \left(0\cdot\cos 0 + 0\cdot\sin(0)\right)$$

$$= \frac{\pi}{2}(-1) + 1 - 0 = 1 - \frac{\pi}{2}$$

Independence of Paths for Line integrals of conservative vector fields

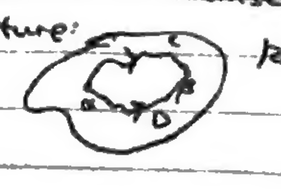
Prop. Suppose C and D are two paths between the same endpoints α and β and suppose $\vec{v}$ is conservative. Then $\int_C \vec{v}\, d\vec{r} = \int_D \vec{v}\, d\vec{r}$

Pf: Apply FTLI: $\int_C \vec{v}\cdot d\vec{r} = f(\beta) - f(\alpha)$ where $v = \nabla f$

$\int_D \vec{v}\cdot d\vec{r} = f(\beta) - f(\alpha) = \int_C \vec{v}\cdot d\vec{r}$

Prop: If $\vec{v}$ satisfies $\int_C \vec{v}\cdot d\vec{r} = \int_D \vec{v}\cdot d\vec{r}$ for all C, D paths between the same endpoints on some open region R and if the components of $\vec{v}$ are all cts on R, then $\vec{v}$ is conservative.

Picture:



pf: Fix any point α in R.
Define $f(\beta) = \int_{\alpha}^{\beta} \vec{v} \cdot d\vec{r}$

$= \int_C \vec{v}\, d\vec{r}$ where C is any curve from α to β

By independence of paths, f is well defined moreover, $\nabla f = \vec{v}$ (exercise, use the FTC) Foundamental Theorem of Calculus

Observation: If $\vec{v}$ is conservative and C is a closed curve (i.e. C starts and ends at the same point), then $\int_C \vec{v} \cdot d\vec{r} = 0$

conversly, if $\int_C \vec{v} \cdot d\vec{r} = 0$ for all closed C, then $\vec{v}$ is conservative

↳ Exercise, (hint: independence of paths)

§ 16.4: Green's Theorem.

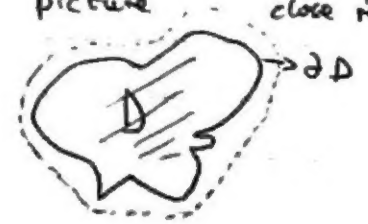
IDEA: In some special cases, line integrals can be computed via double integrals.

Prop (Green's Theorem): Let D be a region in R^2 with a piecewise-smooth boundary curve ∂D

If $P(x,y)$ and $Q(x,y)$ have cts partial derivatives on some open region O containing D, then we have

NB: Because the curve is p.w. smooth and ∂D is "simple close region"

picture

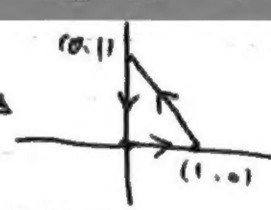


$$\int_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

NB: For this theorem to hold, ∂D needs the positive orientation

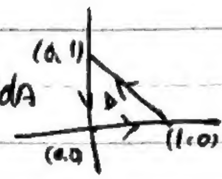
Ex. Compute $\int_C x^4\,dx + xy\,dy$ for C the curve positively oriented around the triangle with vertices $(0,0)$, $(1,0)$ and $(0,$ [illegible]

NB: this would be monsterous normally, b/c the curve is split into 3 pieces



Sol: By Green's Theorem

$$\int_{\partial D} x^4\,dx + xy\,dy = \iint_D \left(\frac{\partial}{\partial x}[xy] - \frac{\partial}{\partial y}[x^4]\right) dA$$



$$= \iint_D y - 0\,dA$$

Note $D = \{(x,y) : 0 \le x \le 1,\ 0 \le y \le 1-x\}$

$$\iint_{\partial D} x^4\,dx + xy\,dy = \iint_D y\,dA$$

$$= \int_{x=0}^{1}\int_{y=0}^{1-x} y\,dy\,dx$$

$$= \int_{x=0}^{1} \frac{1}{2}[y^2]\Big|_{y=0}^{1-x}\,dx$$

$$= \frac{1}{2}\int_{x=0}^{1} ((1-x)^2 - 0)\,dx \qquad \begin{array}{l} u = 1-x \\ du = -dx \end{array}$$

$$= -\frac{1}{2} \cdot \frac{1}{3}(1-x)^3\Big]_{x=0}^{1}$$

$$= -\frac{1}{6}((1-1)^3 - (1-0)^3)$$

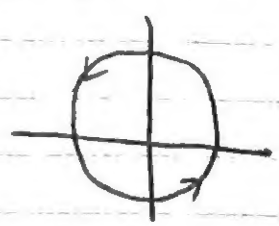
$$= -\frac{1}{6}(-1)$$

$$= \frac{1}{6}$$

Reminder: Green's theorem only works when the curve is a simple closed curve in the plane $\mathbb{R}^2$

Ex: Compute $\int_C (3y - e^{\sin(x)})\,dx + (7x + \sqrt{y^4+1})\,dy$ for C the circle $x^2+y^2=9$

picture



Sol: $\int_{\partial D} (3y - e^{\sin(x)})\,dx + (7x + \sqrt{y^4+1})\,dy$

Green's Theorem $= \iint_D \frac{\partial}{\partial x}[7x + \sqrt{y^4+1}]$

$- \frac{\partial}{\partial y}[3y - e^{\sin(x)}]\,dA$

$$= \iint_D (7-3)\,dA = 4\iint_D dA = 4\,\text{Area}(D) = 4\cdot\pi(3)^2 = 36\pi$$